

Special lines on contact manifolds

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Abstract

In a series of two articles Kebekus studied deformation theory of minimal rational curves on contact Fano manifolds. Such curves are called contact lines. Kebekus proved that a contact line through a general point is necessarily smooth and has a fixed standard splitting type of the restricted tangent bundle. In this paper we study singular contact lines and those with special splitting type. We provide restrictions on the families of such lines, and on contact Fano varieties which have reducible varieties of minimal rational tangents. We also show that the results about singular lines naturally generalise to complex contact manifolds, which are not necessarily Fano, for instance, quasi-projective contact manifolds or contact manifolds of Fujiki class \mathcal{C} . In particular, in many cases the dimension of a family of singular lines is at most 2 less than the dimension of the contact manifold.

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1 Introduction

This article addresses the problem of classification of contact manifolds. A lot of work has already been done in this direction, see [Buc10] for an over-

view and motivation in the projective case. The major remaining task is to classify contact Fano manifolds, which are expected to be always homogeneous spaces. Also complex non-projective contact manifolds recently gained attention, see [HM10], [FP11], [BP12], [PS12], and the classification in the non-projective case is widely open. Even in dimension three, the classification of rationally connected compact contact manifolds with $b_2 \geq 2$ is unknown.

1.1 Special lines on contact Fano manifolds

In the first place, let X be a contact Fano manifold over the complex numbers \mathbb{C} . That is, X is a complex projective manifold of odd dimension $2n+1$, with an ample line bundle L and a twisted nowhere vanishing 1-form $\theta: TX \rightarrow L$ satisfying the following property. Let F be the kernel of θ , so that we have the short exact sequence

$$0 \rightarrow F \rightarrow TX \xrightarrow{\theta} L \rightarrow 0.$$

Then we assume that $d\theta|_{\bigwedge^2 F}$ is nowhere degenerate, that is F_x is a symplectic vector space for each $x \in X$. See Proposition 2.5 for more details.

Among the main tools to approach the problem is the theory of minimal rational curves. In the case of contact Fano manifold X of dimension $2n+1$ it amounts to study *contact lines*. A rational curve C with the normalisation $f: \mathbb{P}^1 \rightarrow C \subset X$ is a *contact line*, if $f^*K_X = \mathcal{O}_{\mathbb{P}^1}(-n-1)$. Here K_X denotes the canonical divisor of X . Unless $X \simeq \mathbb{P}^{2n+1}$, those contact lines exist, cover X and form a family of pure dimension $3n-1$.

For a general contact line with a parametrisation $f: \mathbb{P}^1 \rightarrow X$ the pullback of the tangent bundle TX has a certain standard splitting type, namely:

$$f^*TX \simeq \mathcal{O}^{\oplus(n+1)} \oplus \mathcal{O}(1)^{\oplus(n-1)} \oplus \mathcal{O}(2) = \mathcal{O}(0^{n+1}, 1^{n-1}, 2). \quad (1.1)$$

Contact lines satisfying (1.1) are called standard. Life would be much easier (but more boring), if all the contact lines were known to be standard. This happens for homogeneous contact manifolds (because the group acts transitively on the set of lines). But in general we know very little. Kebekus proved that any line through a general point is smooth and standard. Moreover, an straightforward consequence of results by Kebekus is the following bound for the dimension of the space of singular lines:

Theorem 1.2. *Let X be a contact Fano manifold. The dimension of the scheme parametrising singular contact lines on X is at most $2n-1$ (compared to $3n-1$, the dimension of the Chow variety of contact lines).*

We show how the theorem follows in Subsection 3.1. We elaborate on this bound and generalise it to the case when X is a generically contact manifold, or a contact manifold, which is not necessarily projective (see Subsection 1.2).

Suppose \mathcal{H} is an irreducible component of the Chow variety parametrising contact lines on a contact Fano manifold X . Assume X is not isomorphic to \mathbb{P}^{2n+1} (in this case there is no contact lines, since $L \simeq \mathcal{O}_{\mathbb{P}^{2n+1}}(2)$), and X is not isomorphic to $\mathbb{P}(T^*\mathbb{P}^{n+1})$ (this is the only Fano case with $\mathrm{rk} \mathrm{Pic}(X) \geq 2$). Let \mathcal{H}_x be the set of lines through a fixed point $x \in X$. In this set-up, Kebekus [Kebe05, Thm 3.1] claimed that for general $x \in X$ the set \mathcal{H}_x is irreducible. Unfortunately, there has been a gap in his proof, see [Bucz10, Rem. 3.2] for more details. His argument, in fact, shows the following:

Theorem 1.3 ([Kebe05]). *Suppose X is a contact Fano manifold with $\mathrm{Pic} X$ generated by the class of the quotient line bundle $L = TX/F$. If the set \mathcal{H}_x of lines through a general point $x \in X$ is reducible, then there exists a subset $\mathcal{B} \subset \mathcal{H}$ of non-standard lines (i.e. lines for which (1.1) does not hold), such that $\dim \mathcal{B} = 3n - 2$, i.e. \mathcal{B} is of codimension 1 in \mathcal{H} .*

That is if \mathcal{H}_x is reducible, then there is a lot of non-standard lines. The main result of this article is a description of the potential contact Fano manifold with a lot of non-standard lines.

Theorem 1.4. *Let X be a contact Fano manifold of dimension $2n + 1$. Suppose $\mathcal{B} \subset \mathcal{H}$ is an irreducible subset with $\dim \mathcal{B} = 3n - 2$, consisting only of non-standard lines. Let $B \subset X$ be the locus of those lines. Then:*

- (i) *B is a non-normal irreducible divisor on X . Denote by $\xi: \mathcal{U} \rightarrow B$ be the normalisation map.*
- (ii) *There exists a normal variety \mathcal{R} and a vector bundle E on \mathcal{R} , such that $\mathcal{U} \simeq \mathbb{P}(E) \xrightarrow{\pi} \mathcal{R}$, and $E = (\pi_* \xi^* L)^*$ is a rank $n + 1$ vector bundle on a normal variety \mathcal{R} .*
- (iii) *The restriction $\xi|_{\mathbb{P}^n}$ of the normalisation to any fibre of $\pi: \mathcal{U} \rightarrow \mathcal{R}$ is the normalisation of the image $\xi(\mathbb{P}^n)$ and $(\xi|_{\mathbb{P}^n})^* L = \mathcal{O}_{\mathbb{P}^n}(1)$.*

Part (i) was proved by Kebekus. The rest of the theorem is proved in Section 5.

This article is among the first attempts to study minimal rational curves on a projective manifold X , without assuming they are general, or they pass through a general point.

1.2 Lines on non-projective contact manifolds

More generally, let X be a complex manifold. For some statements below, we do need some minimal assumptions on the parameter spaces, for instance, that the local geometry of singular rational curves determines its global properties. For instance, the dimension of the closure of the locus swept out by the singular rational curves is determined by their infinitesimal deformations. This is guaranteed if X is either quasi-projective or X is compact in *Fujiki class \mathcal{C}* : we say X is Fujiki class \mathcal{C} , if it is bimeromorphic to a compact Kähler manifold [Fuji79]. The parameter space we are mainly interested in is the Barlet space [Barl75], which is the complex geometry analogue of the Chow variety [Koll96, Section I.3].

We assume in addition X has a contact structure, or X is a generically contact manifold. By the latter we mean, that there is a vector subbundle $F \subset TX$, and the quotient line bundle $L = TX/F$, such that $F|_U$ and $L|_U$ determine a contact structure on an open dense subset $U \subset X$, see Subsection 2.5 for more details.

In this setting we can measure the degree of rational curves using the intersection with L . In particular, the contact lines are the (complete) rational curves $C \subset X$ with the intersection number $L.C = 1$. Contrary to the Fano case, there is no guarantee that the lines exist, and the intersection $L.C$ potentially may be zero or negative. This is a major issue, however, assuming the lines do exist, we obtain many results that are parallel to the projective case.

The first proposition generalises Theorem 1.2.

Proposition 1.5. *Suppose X is either a quasi-projective manifold or a complex compact manifold in class \mathcal{C} . Suppose moreover (X, F) is a generically contact manifold of dimension $2n + 1$. Then:*

- (i) *Singular contact lines do not cover X , or in other words, any contact line through a general point (if exists) is smooth.*
- (ii) *If in addition X is projective and L is ample, then the dimension of the space parametrising the singular lines is at most $2n - 1 = \dim X - 2$.*
- (iii) *If (X, F) is contact everywhere and a family of singular lines sweeps out a locus of codimension 1 in X , then the dimension of this family is $2n - 1$.*

We show the proposition in Subsection 3.3, Corollaries 3.4 and 3.5. The second proposition generalises [Kebe01, Lemma 3.5].

Proposition 1.6. *Suppose (X, F) is either a quasi-projective contact manifold or a complex contact compact manifold in class \mathcal{C} . Any line through a general point $x \in X$ (if such line exists) is standard, i.e. if $f: \mathbb{P}^1 \rightarrow X$ is the normalisation of this line, then $f^*TX \simeq \mathcal{O}(0^{n+1}, 1^{n-1}, 2)$.*

We show the proposition in Subsection 4, Corollary 4.5.

1.3 Sketch of proofs and intermediate results

Theorem 1.2 is a combination of two statements, [Kebe01, Prop. 3.3] and [Kebe02, Thm 3.3(2)], see Section 3.1 for details. Its generalisation Proposition 1.5 relies on a technical Proposition 3.3, whose proof follows the method of [Kebe01, Prop. 3.3]. Proposition 1.6 is proved by a standard analysis of possible splitting types of the tangent bundle restricted to lines. See Corollary 4.5 for a stronger version of this statement.

The proof of Theorem 1.4 is more tricky and it is the main new contribution of this article. It is centred about the concept of a *linear subspace* in the contact manifold, which generalises the notion of contact line to higher dimensions. Precisely, a subvariety $\Gamma \subset X$ is a linear subspace, if and only if the normalisation of Γ is a projective space and the restriction of L to Γ is a line bundle of degree 1, see §2.1 for more details.

To show Theorem 1.4, we suppose that there is a component \mathcal{B} of the set of non-standard lines of dimension $3n - 2$. In particular, by Theorem 1.2, a general element of \mathcal{B} is a smooth rational curve. By results of Kebekus, there is a divisor B on X swept out by the lines from \mathcal{B} (see Lemma 5.1). The main aim is to prove that B is dominated by a family of linear subspaces of dimension n .

To construct the linear subspaces, we use the following characterisation of the projective space:

Theorem 1.7. *Suppose Γ is a projective variety with an ample line bundle L , such that a general pair of points $x, y \in \Gamma$ is connected by a single rational curve $f: \mathbb{P}^1 \rightarrow \Gamma$ of degree 1, i.e. $f^*L \simeq \mathcal{O}_{\mathbb{P}^1}(1)$, $x, y \in f(\mathbb{P}^1)$. Then Γ admits a normalisation $\mu: \mathbb{P}^k \rightarrow \Gamma$, where $k = \dim \Gamma$ and $\mu^*L = \mathcal{O}_{\mathbb{P}^k}(1)$.*

This theorem is a consequence of [Kebe02, Thm 3.6], see Section 2.2 for details.

Theorem 1.7 is used twice. In the first place we construct a large family of linear subspaces of dimension 2, next we bundle together the planes, to obtain a family of linear subspaces of dimension n . More precisely, the tangent spaces to B naturally determine a distribution G of rank 1, i.e. a line subbundle of TB defined on an open dense subset of $U \subset B$. Suppose $c \in \mathcal{B}$

is a general non-standard line and $C \subset X$ is the corresponding curve in X . We take the union of leaves of G through points of C , and we let $\overline{\Gamma_c}$ to be the Zariski closure of this union. Then we show in Lemma 5.2 that every two points in $\overline{\Gamma_c}$ are connected by a contact line. Thus the normalisation of $\overline{\Gamma_c}$ is a projective space \mathbb{P}^k . We carefully study the distribution G restricted to $\overline{\Gamma_c}$ and conclude using Lemma 2.3 that the leaves of G actually are lines from \mathcal{B} . In particular, $\dim \overline{\Gamma_c} = 2$, and its normalisation is \mathbb{P}^2 .

This construction also equips each \mathbb{P}^2 with a distinguished point y , and its image in X . We consider $Y \subset X$ to be the union of all distinguished points in X obtained by varying $c \in \mathcal{B}$. The critical step in the proof is the dimension count: we show $\dim Y = n$, see Lemma 5.5. The conclusion is that there is a lot of surfaces $\overline{\Gamma_c}$, with the same distinguished point y . On the other hand the locus P^y of these projective planes is always contained in the locus of lines through a fixed point y , which is known to have dimension at most n . We use these informations to show that general two points x_1, x_2 in P^y are contained in a single $\overline{\Gamma_c}$, whose distinguished point is y . The line in the plane \mathbb{P}^2 normalising $\overline{\Gamma_c}$ is the required line connecting x_1 with x_2 . Thus P^y is normalised by a projective space, and its dimension is calculated to be n . This is the way to construct the family of linear subspaces of dimension n , whose locus is the divisor B .

We also show that there is unique such linear space through a general point of B , and only a finite number of them through any point of B . This is used to compare the family with the normalisation of B . Finally, we conclude using [AD12, Prop. 4.10], which characterises projective space bundles over normal varieties, by analogy to the Fujita's characterisations for bundles on smooth varieties.

1.4 Overview

In Section 2 we provide precise definitions of the notions used in the article, and we review some standard material with its easy adaptation to the content of the article. In Section 3 we investigate singular lines on polarised manifolds. In Section 4 we analyse the possible splitting types of the tangent bundle restricted to contact lines. In Section 5 we show that if there are many non-standard lines on a contact Fano manifolds, then their configuration must be very special.

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2 Preliminaries

Throughout the article suppose X is a complex analytic set, polarised by some line bundle L . We will often assume X is projective manifold and L is ample, but for some of the statements below it is not necessary. Moreover allowing X to have singularities, we obtain some statements valid for subvarieties $Y \subset X$ polarised by $L|_Y$.

2.1 Lines and linear subspaces of a polarised analytic set

We consider a complete rational curve $C \subset X$ with the normalisation

$$f: \mathbb{P}^1 \rightarrow C,$$

and such that $f^*L = \mathcal{O}_{\mathbb{P}^1}(1)$. We say that such C is a *line* on (X, L) — if X is projective and L is very ample, then such curves are just ordinary lines in the projective embedding of X by the complete linear system of L .

Analogously, a *linear subspace* of (X, L) is a subvariety $\Gamma \subset X$, such that the normalisation of Γ is $\mu: \mathbb{P}^k \rightarrow \Gamma$, and $\mu^*L = \mathcal{O}_{\mathbb{P}^k}(1)$, where $k = \dim \Gamma$. Thus, if $k = 1$, then Γ is a line in the above sense.

A *family of linear subspaces of dimension k* is a \mathbb{P}^k -fibration $\pi: \mathcal{U}_{\mathcal{R}} \rightarrow \mathcal{R}$, together with a map $\xi: \mathcal{U}_{\mathcal{R}} \rightarrow X$, such that all the images in X of fibres of π are linear subspaces and $\xi|_{\mathbb{P}^k}: \mathbb{P}^k \rightarrow X$ is the normalisation map of the image. The map ξ is called the *evaluation map*. If $k = 1$, we simply say a *family of lines*, rather than a family of linear subspaces of dimension 1.

We say that two points $x, y \in X$ are connected by a line, if there exists a single line $C \subset X$, such that $x, y \in C$.

For the rest of this subsection we suppose X is in addition projective and L is ample. In this case, the scheme parametrising lines or linear subspaces is also projective. Informally, to see that it is compact, it is enough to observe, that a line cannot degenerate to anything that consists of several components, as the degree of L on each component must be at least 1. More precisely, the lines correspond to a closed subscheme of the Chow variety of X .

In particular, the property of being connected by a line is Zariski closed, that is, the set of pairs $(x, y) \in X \times X$, such that x and y are connected by a line is Zariski closed in $X \times X$.

We also have the standard consequence of the Mori's Bend and Break Lemma:

Lemma 2.1. *Suppose we have a positive dimensional family of distinct lines on X . If X is projective and L is ample, then:*

- *the family may have at most 1 common point;*
- *if there is a point $x \in X$ common to all the lines in the family, the family is proper, and all the lines are smooth at x (or with at worst nodal singularity), then there is at most finitely many lines in this family, with a fixed tangent direction at x .*

Proof. The argument is classical: a positive dimensional family of rational curves through two fixed points must “break” (see [Koll96, Cor. 5.5.2]). But the “broken curve” is an effective, but not primitive cycle, and must be numerically equivalent to the lines in the family. In the case of lines, each primitive curve must intersect with L in degree at least 1, which is a contradiction, as the broken curve needs at least two primitive components, hence the intersection with L is at least 2, which contradicts the numerical equivalence with lines.

The case with the common tangent direction is treated in the same way, but taking into account [Kebe02, Thm 2.4(i)]. \square

2.2 Projective varieties covered by many lines

As outlined in §1.3, in the course of the proof of Theorem 1.4 we will construct several Zariski closed subsets Γ of a contact manifold, such that any two points $x, y \in \Gamma$ are connected by a line contained in Γ . Theorem 1.7 says that such Γ must have the normalisation equal to \mathbb{P}^k , which we now rephrase in the language of §2.1.

Theorem 2.2. *Suppose X is a projective variety with an ample line bundle L , such that two general points $x, y \in X$ are connected by a single line. Then X is a linear subspace of itself, i.e. it admits a normalisation $\mu: \mathbb{P}^k \rightarrow X$, where $k = \dim X$ and $\mu^*L = \mathcal{O}_{\mathbb{P}^k}(1)$.*

This theorem is an easy consequence of a characterisation of projective space by Kebekus [Kebe02, Thm 3.6], but takes into an account a possibility that X is not normal.

Proof. First let us reduce to the case X is normal. So let $\mu: X' \rightarrow X$ be the normalisation, and pick two general points $x', y' \in X'$. Their images $\mu(x')$ and $\mu(y')$ are connected by a line $C \subset X$. Let C' be the proper transform of C . Then $\mu|_{C'}: C' \rightarrow C$ is birational and C is also a rational curve, and the normalisation of $f: \mathbb{P}^1 \rightarrow C$ factorises through C' . So the degree of C' with respect to μ^*L is also 1. Thus it is sufficient to prove the theorem for normal X .

The set of such curves in the Chow variety of X is closed, so also special points x and y are connected by curves of degree 1. Then this is a special case of [Kebe02, Thm 3.6] with L replaced with $L^{\otimes 2}$. \square

2.3 Distributions

In this subsection we summarise some basic material about distributions. We essentially follow the convention of [HM04, §2], where a distribution is an equivalence class of a subbundle defined on some open subset.

Suppose E is a vector bundle on X . We say G is a *distribution* in E if there exists an open dense subset $U \subset X$ and $G \subset E|_U$ is a vector subbundle. Strictly speaking, a distribution is an equivalence class of pairs (G, U) , where (G, U) and (G', U') are equivalent if and only if the two subbundles agree on the intersection $G|_{U \cap U'} = G'|_{U \cap U'}$. For simplicity, we will just denote the distribution by $G \subset E$. The *rank* of G is the rank of G as the vector bundle on U . We let $U(G)$ be the maximal open (dense) subset of X such that (some equivalence of) G is a vector subbundle of $E|_{U(G)}$. Note that if X is normal, then $X \setminus U(G)$ is always of codimension at least 2 in X . In particular, if X is a smooth curve, then G is always a vector subbundle of E .

The word “distribution” is usually associated with a vector subbundle of a tangent bundle. In this article we need a slightly more general situation: we will also consider distributions (for example) in the restriction of the tangent bundle $TX|_Y$ or in the normal bundle $N_{Y \subset X}$, where Y is a closed subvariety of X .

All sorts of natural operations can be applied to distributions. If G_1 and G_2 are distributions in E , then $G_1 + G_2$ and $G_1 \cap G_2$ also are. If $f: Z \rightarrow X$ is a morphism, and the image of f intersects $U(G)$, then f^*G is a distribution in f^*E , etc. We say $G_1 \subset G_2$, if the inclusion holds over $U(G_1) \cap U(G_2)$.

One of the situations we will often consider is when $Y \subset X$ is a subvariety or an analytic subspace, and the distribution is $G \subset TX|_Y$. For example, the tangent bundle $TY \subset TX|_Y$ is a distribution (note that Y needs not to be smooth, $U(TY)$ is the smooth locus of Y). We will say that a subvariety, or an analytic subset $Z \subset Y$ is *G-integral*, if Z intersects $U(G)$ and $TZ \subset G|_Z$ as distributions in $TX|_Z$. We say Z is a *leaf* of G , if Z is G -integral and

$\dim Z = \operatorname{rk} G$.

Lemma 2.3. *Suppose $G \subset T\mathbb{P}^k$ is a rank 1 distribution on a projective space \mathbb{P}^k , which after the restriction to a general line $\mathbb{P}^1 \subset \mathbb{P}^k$ is*

$$\mathcal{O}_{\mathbb{P}^1}(1) \subset T\mathbb{P}^k|_{\mathbb{P}^1} \simeq \mathcal{O}(1)^{\oplus(k-1)} \oplus \mathcal{O}(2) = \mathcal{O}(1^{k-1}, 2).$$

Then there exists a point $y \in \mathbb{P}^k$ such that all lines through y are tangent to G , i.e. the leaves of G are those lines. In particular, the leaves of G are algebraic.

Proof. Suppose $\mathbb{P}^k = \mathbb{P}V$ for a vector space $V \simeq \mathbb{C}^{k+1}$. Consider the Euler sequence for \mathbb{P}^k , restricted to the general line:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^1} \rightarrow \mathcal{O}(1) \otimes V \xrightarrow{\alpha} T\mathbb{P}^k|_{\mathbb{P}^1} \rightarrow 0.$$

The preimage of $G|_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(1)$ in $\mathcal{O}(1) \otimes V$ is a vector bundle of rank 2 on \mathbb{P}^1 isomorphic to $\mathcal{O}_{\mathbb{P}^1}(0, 1)$. Such a bundle contains a unique subbundle of rank 1 isomorphic to $\mathcal{O}(1)$. Thus G determines a line $\hat{y} \subset V$ such that

$$\alpha^{-1}(G) = \mathcal{O} \oplus \mathcal{O}(1) \otimes \hat{y} \subset \mathcal{O}(1) \otimes V.$$

Let $y \in \mathbb{P}V$ be the corresponding point in the projective space. The above equality explains that G is tangent to lines through y at all points of the fixed general \mathbb{P}^1 .

Apriori, the point y could vary when we vary the line. To exclude this possibility, let $U \subset \operatorname{Gr}(\mathbb{P}^1, \mathbb{P}^k)$ be an open subset of lines in \mathbb{P}^k , such that for $\ell \in U$, the assumption of the lemma is satisfied, i.e. $G|_{\ell} \simeq \mathcal{O}_{\ell}(1)$. Then we have a map $\phi: U \rightarrow \mathbb{P}^k$, mapping ℓ to the point $y = \phi(\ell)$ constructed as above. Suppose two lines $\ell, \ell' \in U$ intersect in a point $x \in \mathbb{P}^k$ and G is defined at x . Then $x, \phi(\ell), \phi(\ell')$ are on a line, because both $\phi(\ell)$ and $\phi(\ell')$ are on the line passing through x and tangent to G at x . In other words, ϕ maps the pencil of lines passing through a general point x into a line. Moreover, we claim this line is tangent to G .

To see that, consider a general plane \mathbb{P}^2 containing x and y , and let $W \simeq \mathbb{P}^1$, $W \subset \operatorname{Gr}(\mathbb{P}^1, \mathbb{P}^k)$ be the set of lines contained in this \mathbb{P}^2 and passing through x . A general line $\ell \in W$ belongs to U . Thus ϕ restricts to a rational map $W \dashrightarrow \langle x, y \rangle$, which extends to a regular map $\xi: \mathbb{P}^1 \rightarrow \mathbb{P}^1$. Pick a general point $z \in \langle x, y \rangle$. Since G is defined at x , it is also defined at z . Consider a curve $z_t \in \mathbb{P}^2$, with $z_0 = z$, and let $\ell_t \in W$ be the line spanned by x and z_t . Then $\lim_{t \rightarrow 0} \phi(\ell_t) = \xi(\ell_0) = y \neq z$. G_{z_t} for general t is tangent to $\langle z_t, \phi(\ell_t) \rangle$. Taking the limit we get that G_z is the tangent direction to the line $\langle z, \phi(\ell_0) \rangle = \langle x, z \rangle$, which proves the claim.

The claim above proves, that if ℓ is a general line, and $x \in \ell$ is a general point, then the line $\langle x, \phi(\ell) \rangle$ is tangent to G . In particular, for all lines contained in the plane spanned by ℓ and $\phi(\ell)$, the map ϕ is constant and equal $\phi(\ell)$. If $k = 2$, then the lemma is proved, as the plane is just equal to \mathbb{P}^k . In general, suppose $y = \phi(\ell)$ is a general point of the image of ϕ . Then G is not defined at y , because at least one dimensional family of lines through y is tangent to G . Thus there are at most finitely many points of the image $\phi(U)$ on the line $\langle x, y \rangle$. But a general line through x is mapped to some point on the fixed line $\langle x, y \rangle$, the integral line of G through x . Thus the image of such general line must be one of those finitely many points, and by continuity there can be only one point which is in the image of lines through x , namely y .

Summarising, let $z \in \mathbb{P}^k$ be another general point. The lemma claims that G_z is the tangent direction to the line $\langle y, z \rangle$. Indeed, the line $\langle x, z \rangle$ is in U and is mapped to y . Therefore for every point in $\langle x, z \rangle$ where G is defined, G at this point is tangent to the line through y . \square

2.4 Manifolds with corank 1 distributions

Suppose X is a complex manifold and $F \subset TX$ is a distribution, such that $U(F) = X$, i.e. a distribution defined on whole of X . Suppose $\text{rk } F = \dim X - 1$ and let $\theta : TX \rightarrow TX/F =: L$ be the quotient map, so that the following is a short exact sequence of vector bundles on X :

$$0 \rightarrow F \rightarrow TX \xrightarrow{\theta} L \rightarrow 0.$$

In this situation we say that (X, F) is a *manifold with a corank 1 distribution*.

Observation 2.4. Suppose (X, F) is a manifold with a corank 1 distribution, and $Y \subset X$ is an analytic subset. Let Y_0 be the smooth locus of Y and consider a distribution $G \subset TY_0$, which is defined as $G := TY_0 \cap F$. Then either Y is F -integral, or there exists an open dense subset $Y' \subset Y$, such that $(Y', G|_{Y'})$ is a manifold with a corank 1 distribution.

The above observation captures the motivation for our treatment of manifolds with corank 1 distributions. That is, even though our primary interest is in contact manifolds (see §2.5), in our arguments we will prove claims about subvarieties of contact manifolds, and they have the property of being (generically) manifolds with corank 1 distributions. As a side result, some of our intermediate results apply to a more general situation, than just contact manifolds.

Proposition 2.5 ([Bucz09, Prop C.1(i) and (iv)]). *With the assumptions as above:*

(i) $d\theta$ determines a well defined homomorphism of vector bundles:

$$d\theta : \bigwedge^2 F \rightarrow L.$$

(ii) If $\Delta \subset X$ is F -integral, Δ_0 is the smooth locus of Δ , then $d\theta|_{\Delta_0} \equiv 0$.
In particular:

$$\dim \Delta \leq \operatorname{rk} F - \frac{1}{2} \min_{x \in \Delta} (\operatorname{rk} d\theta_x).$$

Suppose $Y \subset X$ is a subvariety, $G \subset TX|_Y$ is another distribution (not necessarily defined on all of Y). In this situation by G^{\perp_F} we denote the distribution in $TX|_Y$:

$$G^{\perp_F} := (G \cap F|_Y)^{\perp_{d\theta}} \subset F|_Y. \quad (2.6)$$

This distribution is defined on an open dense subset of Y where the rank of $G \cap F|_Y$ is constant and where $\operatorname{rk} d\theta$ is constant.

Consider the open subset $X_0 \subset X$, where $\operatorname{rk} d\theta_x$ is constant and equal to $2r$ for $x \in X_0$. Suppose $\Delta \subset X_0$ is an analytic submanifold. We say Δ is *maximally F -integral* if Δ is F -integral and $\dim \Delta = \dim X - r - 1$. This is an analogue of a Legendrian subvariety in contact manifold, but here F needs not to be a contact structure on X (that is, $d\theta_x$ is not necessarily non-degenerate).

Lemma 2.7. *Pick $x \in X_0$ and let $v \in T_x X$. Then $v \in (F_x)^{\perp_{d\theta}}$ (i.e. v is in the degeneracy locus of $d\theta$), if and only if $v \in T_x \Delta$ for every maximally F -integral Δ containing x .*

Proof. Clearly $(F_x)^{\perp_{d\theta}}$ is the null locus of $d\theta|_{F_x}$, so $T_x \Delta$ must contain it for every maximally G -integral Δ .

We now prove the other direction. The problem is analytically local around x , so we can assume X is an analytically open subset of $\mathbb{C}^{\dim X}$, $x = 0$, and θ is in the Darboux normal form $\theta = dx_0 - \sum_{i=1}^r x_i dx_{r+i}$. Clearly $F_x = \{dx_0 = 0\}$ and $(F_x)^{\perp_{d\theta}} = \{dx_0 = \dots = dx_{2r} = 0\}$. Let $\Delta_1 = \{x_0 = x_1 = \dots = x_r = 0\}$ and $\Delta_2 = \{x_0 = x_{r+1} = \dots = x_{2r} = 0\}$. These are maximally F -integral containing x , so

$$\begin{aligned} v \in T_0 \Delta_1 \cap T_0 \Delta_2 &= \\ \{dx_0 = dx_1 = \dots = dx_r = 0\} \cap \{dx_0 = dx_{r+1} = \dots = dx_{2r} = 0\} &= \\ &= (F_x)^{\perp_{d\theta}} \end{aligned}$$

□

All the manifolds with corank 1 distributions come with a natural polarisation in the sense of §2.1. Namely, $(X, L = TX/F)$ is a polarised manifold.

Lemma 2.8. *If $\Gamma \subset X$ is a linear subspace of (X, L) , then Γ is F -integral. In particular, lines are always F -integral.*

□

2.5 Contact manifolds

Let (X, F) be a manifold with a corank 1 distribution, with the short exact sequence $0 \rightarrow F \rightarrow TX \xrightarrow{\theta} L \rightarrow 0$. We say that (X, F) is a contact manifold, if $d\theta: \bigwedge^2 F \rightarrow L$ as in Proposition 2.5 is nowhere degenerate. That is, $d\theta$ makes F_x into a symplectic vector space for each $x \in X$. In particular, $\dim X$ is odd.

If X is a contact manifold of dimension $2n + 1$, then $-K_X$ is a divisor linearly equivalent to the Cartier divisor of the line bundle $L^{\otimes(n+1)}$. Our main interest is when X is projective, in fact Fano, which is therefore equivalent to L being ample. However, some of the statements are true in a more general setting. First let us recall, the major classification result:

Theorem 2.9 ([KPSW00, Dema02]). *Suppose X is a projective contact manifold. Then either $X = \mathbb{P}(T^*M)$ for some projective manifold M and $L \simeq \mathcal{O}_{\mathbb{P}(T^*M)}(1)$, or X is Fano and $\text{Pic } X = \mathbb{Z}[L]$, or $X = \mathbb{P}^{2n+1}$ and $L = \mathcal{O}_{\mathbb{P}^{2n+1}}(2)$.*

Also the classification of contact Fano manifolds is known in low dimension.

Theorem 2.10. *Suppose X is a contact Fano manifold with the Picard group generated by L , i.e. $\text{Pic } X = \mathbb{Z}[L]$. Let $\dim X = 2n + 1$. Then $n \geq 2$ and if $n = 2$, then X is the five dimensional homogeneous G_2 -manifold.*

Excluding the case $n = 1$ has been claimed first by Ye [Ye94], but his argument contains a gap. In fact he only provides a proper argument for Theorem 2.9 in the case $n = 1$. However nowadays, it is not difficult to treat the missing case, there are at least two approaches. Firstly, there are not that many Fano threefolds with Picard number 1 and index 2, and one can just check all the possibilities. Alternatively, one can use Hirzebruch-Riemann-Roch theorem, and use cohomological criterions for a manifold to be a projective space. This latter approach has been implemented in [BP12] in a more general situation.

The case $n = 2$ has been proved by Druel [Drue98] using results of [Muka89], [Mell99], and [Beau98].

Maximally integral submanifolds (or subvarieties) of contact manifolds are called *Legendrian*. That is, a subvariety, or an analytic subspace $Y \subset X$ is Legendrian, if it is of pure dimension n and $TY \subset F|_Y$ (as distributions in $TX|_Y$).

2.6 Parameter spaces for lines on contact Fano manifolds

Let X be a contact Fano manifold of dimension $2n + 1$. A lot of attention aims to understand lines on X . Let us underline, that we mean lines with respect to the polarisation L , as in §2.1. In particular, if $X = \mathbb{P}^{2n+1}$, then $L \simeq \mathcal{O}_{\mathbb{P}^{2n+1}}(2)$ and there are no lines on X . In all the other projective cases, the lines exist and cover X : for $X = \mathbb{P}(T^*M)$, the ordinary lines in the fibres are lines with respect to L ; for X Fano with $\text{Pic } X = \mathbb{Z}[L]$, this is observed (for example) in [Kebe01, §2.3].

Let \mathcal{H} be an irreducible component of the Chow variety of X , whose members represent lines in (X, L) . Thus each point in $c \in \mathcal{H}$ represents an irreducible rational curve $C \subset X$ with the normalisation $f: \mathbb{P}^1 \rightarrow C$, and $f^*L = \mathcal{O}_{\mathbb{P}^1}(1)$.

From the definition of the Chow variety, \mathcal{H} comes with the following diagram:

$$\begin{array}{ccc} & \mathcal{U}_{\mathcal{H}} & \\ \pi \swarrow & & \searrow \phi \\ \mathcal{H} & & X \end{array}$$

where $\mathcal{U}_{\mathcal{H}}$ is the *universal family*, that is the subscheme of $\mathcal{H} \times X$, such that the projection $\pi: \mathcal{U}_{\mathcal{H}} \rightarrow \mathcal{H}$ is equidimensional and the fibre $\pi^{-1}(c)$ is $\{c\} \times C$, C is the curve corresponding to c . The map $\phi: \mathcal{U}_{\mathcal{H}} \rightarrow X$ is a projection on the second coordinate.

For $x \in X$, we let \mathcal{H}_x be the “scheme of lines through x ”, that is $\pi(\phi^{-1}(x))$. We also let $H_x \subset X$ be the “union of lines through x ”, that is $\phi(\pi^{-1}(\mathcal{H}_x))$.

Remark 2.11. Kebekus in his presentation of [Kebe01], [Kebe05] assumes in addition that \mathcal{H} *dominates* X , i.e. there exists a line from \mathcal{H} that passes through a general point of X . However, this assumption is redundant, and below we briefly explain why.

Proposition 2.12 ([Kebe01, Prop. 4.1]). *Suppose X is a contact Fano manifold of dimension $2n + 1$, not isomorphic to \mathbb{P}^{2n+1} . Let $x \in X$ be any point.*

Then the reduced scheme $(H_x)_{\text{red}} \subset X$ is a Legendrian subvariety. In particular, it is of pure dimension n . Furthermore, since there can be at most finitely many lines through two points of X (by Lemma 2.1), it follows, that \mathcal{H}_x is of pure dimension $n - 1$.

Note that the proof of [Kebe01, Prop. 4.1] does not use the assumption that \mathcal{H} dominates X .

Corollary 2.13. *Suppose X is a contact Fano manifold of dimension $2n + 1$, not isomorphic to \mathbb{P}^{2n+1} . Then $\dim \mathcal{H} = 3n - 1$, and \mathcal{H} dominates X .*

Proof. As in [Kebe01, (2.1)], $\dim \mathcal{H} \geq 3n - 1$. On the other hand

$$\dim \mathcal{H} = \dim \mathcal{H}_x + \dim H - 1,$$

where $H \subset X$ is the locus of \mathcal{H} (i.e. the union of all lines from \mathcal{H}), and $x \in H$ is a general point. Since $\dim \mathcal{H}_x = n - 1$ and $\dim H \leq \dim X = 2n + 1$, we must have $\dim \mathcal{H} = 3n - 1$ and $H = X$. \square

Similarly, if $\mathcal{B}, \mathcal{S} \subset \mathcal{H}$ are some subfamilies of lines, then we define $\mathcal{U}_{\mathcal{B}}, \mathcal{U}_{\mathcal{S}} \subset \mathcal{U}_{\mathcal{H}}$, $\mathcal{B}_x, \mathcal{S}_x \subset \mathcal{H}_x$, $B_x, S_x \subset H_x$ in the analogous way. Typically, \mathcal{B} will be the family of non-standard lines, and \mathcal{S} will be the family of singular lines, or rather they will be some irreducible components of these families.

3 Singular lines

Suppose X is a complex manifold, and L is a line bundle on X . Consider the Barlet space of X , i.e. the complex geometry analogue of the Chow variety of X . Inside the Barlet space consider an irreducible component \mathcal{S} of the subset parametrising singular lines on (X, L) . Let $\mathcal{U}_{\mathcal{S}} \subset \mathcal{S} \times X$ be the universal family for \mathcal{S} and let $S \subset X$ be the image of the projection $\mathcal{U}_{\mathcal{S}} \rightarrow X$. That is, S is the locus of lines from \mathcal{S} .

3.1 Kebekus results on singular lines in the projective case

Deformations of singular rational curves are studied by Kebekus in [Kebe02] (in general setting) and in [Kebe01, §3] (in the setting of contact Fano manifolds). The summary of these results (restricted to the setting of lines) is presented in the following proposition.

Proposition 3.1. *Let (X, L) be a polarised projective variety (not necessarily smooth, or normal). Suppose L is ample and $x \in X$ is a general point. Then the set of singular lines through x is at most finite, and all these lines are smooth at x . If X is in addition a contact Fano manifold with the contact distribution $F \subset TX$ and the quotient $TX/F \simeq L$, then all lines through x are smooth.*

For a proof see [Kebe02, Thm 3.3(2)] and [Kebe01, Prop. 3.3]. We will also generalise the latter result and proof in Proposition 3.3. At the moment we conclude with a proof of Theorem 1.2.

Proof of Theorem 1.2. Let X be a contact Fano manifold with the ample line bundle L , and suppose \mathcal{S} and S are as in the beginning of this section. By Proposition 3.1 applied to X (contact case), $S \neq X$, thus $\dim S \leq \dim X - 1 = 2n$. Further, by Proposition 3.1 applied to $(S, L|_S)$ (general case), there are finitely many singular lines through a general point of S . Thus $\dim \mathcal{S} = \dim S - 1 \leq 2n - 1$ as claimed. \square

3.2 Singular lines as morphisms

From now on we will assume that X is either a quasi-projective manifold, or a complex manifold of Fujiki class \mathcal{C} , i.e., it is bimeromorphic to a compact Kähler manifold.

An integral singular rational curve can be always dominated by an integral singular plane cubic, i.e. by a rational curve with a single node or cusp. For that reason, let Q be a singular plane cubic, and let $\mathrm{Hom}^{lin}(Q \rightarrow X)$ be the space of morphisms $f: Q \rightarrow X$ such that the degree of the line bundle f^*L is 1. Note that such morphism is automatically birational onto its image.

Lemma 3.2. *There exists an irreducible component $\mathrm{Hom}_{\mathcal{S}}$ of the scheme $\mathrm{Hom}^{lin}(Q \rightarrow X)$, which dominates \mathcal{S} , i.e. we have the commutative diagram:*

$$\begin{array}{ccccc}
 & & \mathrm{Hom}_{\mathcal{S}} \times Q & & \\
 & \swarrow pr_1 & \downarrow im_Q & \searrow ev & \\
 \mathrm{Hom}_{\mathcal{S}} & & \mathcal{U}_{\mathcal{S}} & & X \\
 \downarrow im & \swarrow \pi & & \searrow \phi & \\
 \mathcal{S} & & & &
 \end{array}$$

and the map $im: \mathrm{Hom}_{\mathcal{S} \times q} \rightarrow \mathcal{S}$ is dominant. Here $ev(f, q) = f(q)$ is the evaluation map, pr_1 is the projection map, $im(f)$ is the image curve, considered as a point in \mathcal{S} , and $im_Q(f, q) = (im(f), f(q))$. Moreover, fix a smooth point $q \in Q$. Then the map $im_Q|_{\mathrm{Hom}_{\mathcal{S}} \times q}: \mathrm{Hom}_{\mathcal{S}} \times q \rightarrow \mathcal{U}_{\mathcal{S}}$ is dominant too.

Proof. The first claim is clear, since any singular rational curve can be dominated by a rational curve with a single node or single cusp, i.e. by a singular plane cubic. To see that $\mathrm{im}_Q|_{\mathrm{Hom}_S}$ is dominant, we note that the automorphism group of Q acts transitively on the smooth points of Q . Compare also with [Kebe02, Prop. 2.8]. \square

3.3 Singular lines and distribution

Suppose in addition $0 \rightarrow F \rightarrow TX \xrightarrow{\theta} L \rightarrow 0$ is a short exact sequence of vector bundles as in §2.4, so that X is a manifold with a corank 1 distribution.

Define the distribution $G \subset TX|_S$ by $G := TS \cap F$. By Observation 2.4, either S is F -integral, or there exists an open dense subset $S' \subset S$, such that $(S', G|_{S'})$ is a manifold with a corank one distribution.

Proposition 3.3. *With the notation as above, suppose $c \in S$ is a general point corresponding to a singular line $C \subset X$, and let $s \in C$ be a general point on this line. Then C is tangent to the distribution G^{\perp_F} in TS , i.e. TC is contained in the degeneracy subbundle of G with respect to $d\theta|_G$.*

The proof of this proposition follows quite strictly the lines of proof of [Kebe01, Prop. 3.3], however our statement is stronger.

Proof. Since $C \subset S$, if S is F -integral, then the claim follows from Proposition 2.5(ii). Thus, using Observation 2.4, we may suppose that S (generically) is a manifold with a corank 1 distribution.

Pick a singular plane cubic Q and an irreducible component Hom_S as in Lemma 3.2, so that the map $\mathrm{ev}_q : \mathrm{Hom}_S \rightarrow S$, $\mathrm{ev}_q(f) = f(q)$ is dominant for any smooth point q of Q .

For a general morphism $f \in \mathrm{Hom}_S$ the tangent map of ev_q has the maximal rank at f , i.e. $\mathrm{rk}_f T\mathrm{ev}_q = \dim S$. The set of pairs (f, q) for which the rank is maximal is open in $\mathrm{Hom}_S \times Q$. By [Hart77, II.6.10.2, II.6.11.4 and Ex. II.6.7] the smooth points of Q are in 1:1-correspondence with line bundles of degree one. So fix a general $f \in \mathrm{Hom}_S$, and a general point $q \in Q$, such that $\mathcal{O}_Q(q) \not\cong f^*(L)$ and $\mathrm{rk}_f T\mathrm{ev}_q = \dim S$.

Let $s = f(q)$ and $C = f(Q)$. Note C and s are general as requested by the assumptions of the proposition. The claim of the proposition is that $T_s C \subset (G_s)^\perp$. Since $C \subset S$, we always have $T_s C \subset G_s$, and thus the claim is that $T_s C$ is contained in the degeneracy locus of $d\theta|_G$, i.e. $T_s C \subset (G_s)^{\perp_{d\theta|_G}}$. Suppose on the contrary — by Lemma 2.7 applied to the manifold with (generically) corank 1 distribution (S, G) , there exists $\Delta \subset S$, a maximally

G -integral analytic submanifold of S , which is transversal to C at s . Since ev_q has the maximal rank, we can find a section $\Gamma \subset \text{Hom}_S$ over Δ , i.e. a submanifold Γ such that $\text{ev}_q|_\Gamma : \Gamma \rightarrow \Delta$ is biholomorphic near $f \rightarrow s$. By the construction, $\text{ev}(\Gamma \times Q)$ contains a small analytic submanifold Δ' of S of dimension $\dim \Delta + 1$, that contains Δ and $C = f(Q)$. In particular, Δ' cannot be G -integral.

So there exists $(f', q') \in \Gamma \times Q$ in a small neighbourhood of (f, q) , such that $T_{f'(q')} \Delta' \not\subset G_{f'(q')}$, so $\theta(T_{f'(q')} \Delta') \not\equiv 0$. But

$$\begin{aligned} T_{f'(q')} \Delta' &= d \text{ev}(T_{f'} \Gamma + T_{q'} Q) = d \text{ev}_{q'}(T_{f'} \Gamma) + T_{f'(q')} f'(Q) \\ &\subset \{ \sigma(q') \mid \sigma \in H^0((f')^* TX) \} + T_{f'(q')} f'(Q). \end{aligned}$$

Since $\theta(T_{f'(q')} f'(Q)) \cong 0$ by Lemma 2.8, there must exist a section $\sigma \in H^0((f')^* TX)$, such that $(f')^*(\theta)(\sigma)(q') \neq 0$. But $(f')^*(\theta)(\sigma) \in H^0((f')^* L)$ and

$$\sigma(q) \in T_{f'(q)} \Gamma \subset G_{f'(q)}$$

So $(f')^*(\theta)(\sigma) \neq 0$ and it vanishes at q , a contradiction with our choice of q . So $T_s C \subset G_s \cap (G_s)^\perp$ as claimed. \square

We obtain the following corollaries which generalise Proposition 3.1 to the situation, where either X is a projective *generically contact* manifold, or X is a contact manifold, which is not necessarily projective. In the first situation, X could be for instance a birational modification of a projective contact manifold. In the second situation, X could be a quasi-projective contact manifold (see [HM10]) or a class \mathcal{C} contact manifold (see [FP11], [BP12]).

Corollary 3.4. *Suppose (X, F) is a complex manifold of dimension $2n + 1$ with a corank 1 distribution, and the quotient line bundle $L = TX/F$. Assume that (X, F) is generically contact, that is there exists an open dense subset $X_0 \subset X$ such that $(X_0, F|_{X_0})$ is a contact manifold. Let \mathcal{S} and S be as above. Then $S \neq X$, i.e. the singular contact lines do not cover X . If in addition X is projective and L is ample, then $\dim \mathcal{S} \leq 2n - 1 = \dim X - 2$.*

Proof. Suppose \mathcal{S} is non-empty, $c \in \mathcal{S}$ is a general singular line, and $s \in C$ is a general point on this line. Then by Proposition 3.3, one has $T_s S \cap F_s \subset T_s C^\perp \subsetneq F_s$. Thus $T_s S \subsetneq T_s X$ and $S \neq X$.

If X is projective and L is ample, then by Proposition 3.1 (general case applied to S) we get $\dim \mathcal{S} \leq \dim S - 1 \leq \dim X - 2$. \square

Corollary 3.5. *Suppose (X, F) is a complex contact manifold of dimension $2n + 1$. Let \mathcal{S} and S be as above. If $\dim S = \dim X - 1 = 2n$, then $\dim \mathcal{S} = \dim X - 2 = 2n - 1$, that is, if the locus of some singular lines is a divisor in X , then the dimension of the space of those lines is $2n - 1$.*

Proof. Let $S_0 \subset S$ be the smooth locus of S . Since S has codimension 1 in X , hence $G := (TS)^\perp$ is a rank one distribution in TS_0 . And by Proposition 3.3, the singular lines in \mathcal{S} must be the leaves of G . In particular, there is a unique such line through a general point of S , and we must have $\dim \mathcal{S} = \dim S - 1 = \dim X - 2$ as claimed. \square

4 Splitting types on special lines

In this section we suppose that X is a complex contact manifold, with a contact distribution $F \subset TX$ and the quotient line bundle $L = TX/F$. In particular, X does not need to be projective, or compact.

Suppose $f: \mathbb{P}^1 \rightarrow X$ is a holomorphic map such that $f^*L \simeq \mathcal{O}(1)$, that is f is a parametrisation of a line. We consider f^*TX . By [KPSW00, Prop. 2.8] the splitting type of this vector bundle is

$$f^*TX = \bigoplus_{i=1}^n \mathcal{O}(a_i) \oplus \bigoplus_{j=1}^{n+1} \mathcal{O}(-b_j) = \mathcal{O}(a_1, \dots, a_n, -b_1, \dots, -b_{n+1})$$

with $a_i > 0$ and $b_j \geq 0$. We also have $c_1(f^*TX) = n + 1$, so $\sum a_i - \sum b_j = n + 1$. Since the differential gives a non-zero morphism $T\mathbb{P}^1 \simeq \mathcal{O}(2) \rightarrow f^*TX$, we must have at least one $a_i \geq 2$.

Definition 4.1. A line C in (X, L) parametrised by $f: \mathbb{P}^1 \rightarrow X$ is *standard*, if all the integers $b_i = 0$. In such case we must have $a_1 = 2$, and $a_2 = \dots = a_n = 1$, and $f^*TX = \mathcal{O}_{\mathbb{P}^1}(0^{n+1}, 1^{n-1}, 2)$.

Further consider f^*F . Since $F^* \simeq F \otimes L$, we also have:

$$f^*F \simeq \bigoplus_{i=1}^n \mathcal{O}(c_i) \oplus \bigoplus_{i=1}^n \mathcal{O}(1 - c_i) = \mathcal{O}(c_1, \dots, c_n, 1 - c_1, \dots, 1 - c_n)$$

for some integers $c_i > 0$. In particular, there are exactly n strictly positive entries in this splitting.

Lemma 4.2. *The short exact sequence*

$$0 \rightarrow f^*F \rightarrow f^*TX \xrightarrow{f^*\theta} \mathcal{O}(1) \rightarrow 0$$

does not split, and $c_i = a_i$.

Proof. Suppose on the contrary, that the exact sequence splits. Then one of the a_i is equal to 1, say $a_n = 1$. Further

$$f^*F = \mathcal{O}(a_1, \dots, a_{n-1}, -b_1, \dots, -b_{n+1}),$$

a contradiction, since there are only $n - 1$ strictly positive entries in this splitting.

Consider the restriction $f^*\theta: \mathcal{O}(a_1, \dots, a_n) \rightarrow \mathcal{O}(1)$. Since $a_i \geq 1$ and the sequence does not split, this restriction must be identically zero. So the positive part comes from f^*F and $c_i = a_i$. \square

Now suppose there is only one $b = b_n \geq 0$ and the remaining $b_1 = \dots = b_{n-1} = 0$. Equivalently, there is at most one negative term in the splitting of f^*TX . Then one of the a_i , say a_n , must satisfy $1 - a_n \leq -b$. So suppose $a_n = b + c$ for some $c > 0$. We must have $(\sum_{i=1}^{n-1} a_i) + c = n + 1$, and we conclude:

Lemma 4.3. *If there is at most one negative term in the splitting of f^*TX , then either*

$$\begin{aligned} f^*TX &= \mathcal{O}(-b, 0^n, 1^{n-2}, 2, b+1) \text{ and} \\ f^*F &= \mathcal{O}(-b, -1, 0^{n-2}, 1^{n-2}, 2, b+1), \\ &\text{or} \\ f^*TX &= \mathcal{O}(-b, 0^n, 1^{n-1}, b+2) \text{ and} \\ f^*F &= \mathcal{O}(-b-1, 0^{n-1}, 1^{n-1}, b+1) \end{aligned}$$

for some $b \geq 0$. Note that in the second case the image of f must have cuspidal singularities (unless $b = 0$), i.e. the curve is not an immersed curve.

The lemma applies to a situation where there is a set of lines filling in a divisor.

Lemma 4.4. *Suppose \mathcal{B} is an irreducible variety parametrising lines on X , and let $B \subset X$ be the locus of those lines. Assume $c \in \mathcal{B}$ is a general line from \mathcal{B} , and let $C \subset B$ be the corresponding curve, with a birational parametrisation $f: \mathbb{P}^1 \rightarrow C$. Then f^*TX has at most $\text{codim}(B \subset X)$ negative terms in its splitting, and the distribution $f^*TB \subset f^*TX$ is contained in $(f^*TX)^{\geq 0}$.*

Proof. Since C is a rational curve, we can replace \mathcal{B} with $\text{Hom}_{\mathcal{B}}$, a subvariety of $\text{Hom}(\mathbb{P}^1, X)$, consisting of the morphisms, whose images are the curves in \mathcal{B} . The locus of $\text{Hom}_{\mathcal{B}}$, i.e. the union of images of all morphisms from $\text{Hom}_{\mathcal{B}}$, is equal to B . In this situation $f \in \text{Hom}_{\mathcal{B}}$ is a general point, in particular, it

is a smooth point of $\mathrm{Hom}_{\mathcal{B}}$, even though f might be a singular or non-reduced point of $\mathrm{Hom}(\mathbb{P}^1, X)$. Similarly, if $p \in \mathbb{P}^1$ is a general point, then $f(p)$ is a general point in B .

Thus all the tangent directions in $T_f \mathrm{Hom}_{\mathcal{B}} \subset T_f \mathrm{Hom}(\mathbb{P}^1, X)$ can be realised as curves in $\mathrm{Hom}_{\mathcal{B}}$, that is, as deformations of C , which (in particular) are contained in B . We have $T_f \mathrm{Hom}(\mathbb{P}^1, X) = H^0(f^*TX)$ and the differential of the evaluation map $\mathrm{Hom}(\mathbb{P}^1, X) \times \mathbb{P}^1 \rightarrow X$ at (f, p) is the evaluation of sections $H^0(f^*TX) \rightarrow f^*(TX)_p = T_{f(p)}X$ [Koll96, Prop. II.3.4]. The deformations obtained from $\mathrm{Hom}_{\mathcal{B}}$ sweep out B , so the image of the evaluation contains $T_{f(p)}B$, which is only possible if the number of non-negative terms in the splitting of f^*TX is at least $T_{f(p)}B = \dim B$. Equivalently, the number of negative terms is at most $\mathrm{codim}(B \subset X)$. \square

Corollary 4.5. *Suppose \mathcal{B} is an irreducible subset of the Barlet space, whose general member is a non-standard line. Then:*

- (i) *the locus B of \mathcal{B} does not cover X , and*
- (ii) *if, in addition, $\mathrm{codim}(B \subset X) = 1$, then the splittings of f^*TX and f^*F are as in Lemma 4.3, with $b < 0$, where f is the normalisation of the general line from \mathcal{B} .*

5 Divisors of non-standard lines

In this section we assume throughout that X is a contact Fano manifold of dimension $2n + 1$ with a contact distribution $F \subset TX$ and the ample line bundle $L = TX/F$. In addition we assume $X \neq \mathbb{P}^{2n+1}$ and $X \neq \mathbb{P}(T^*\mathbb{P}^{n+1})$, so that $\mathrm{Pic} X = \mathbb{Z}[L]$, see Theorem 2.9. We always have $n \geq 2$, see Theorem 2.10.

In this setting, pick any irreducible component \mathcal{H} of the set parametrising lines in X . If $c \in \mathcal{H}$ is a general line with a birational parametrisation $f: \mathbb{P}^1 \rightarrow X$, then the splitting type of f^*TX is standard:

$$f^*TX = \mathcal{O}(0^{n+1}, 1^{n-1}, 2).$$

As in Definition 4.1, we say that a line $c \in \mathcal{H}$ parametrised by f is *standard* if it has the above splitting type of f^*TX . We say c is *non-standard*, if it has any other splitting of f^*TX . In this section we are interested in the subset of \mathcal{H} consisting of non-standard lines. So suppose throughout this section that $\mathcal{B} \subset \mathcal{H}$ is a closed irreducible subset containing only non-standard lines. Our aim is to analyse the case when the codimension of the set parametrising lines

with non-standard splitting type of TX is 1. Thus we assume throughout this section that $\mathcal{B} \subset \mathcal{H}$ is a prime divisor, that is

$$\dim \mathcal{B} = 3n - 2.$$

Let $B \subset X$ be the locus of all lines in \mathcal{B} . By Corollary 4.5 we must have $B \neq X$. The lemma below explains that the claim of [Kebe05, Prop. 3.2] is equivalent to the claim that \mathcal{B} as above does not exist.

Lemma 5.1. *In the situation as above, if $\text{codim}(\mathcal{B} \subset \mathcal{H}) = 1$, then for all $x \in B$ the set \mathcal{B}_x is a union of irreducible components of \mathcal{H}_x . In particular, $\dim \mathcal{B}_x = n - 1$, and analogously, B_x is a union of irreducible components of H_x and $\dim B_x = n$. Moreover, B is a divisor, i.e. $\dim B = 2n$.*

Proof. \mathcal{H}_x is of pure dimension $n - 1$, and H_x is of pure dimension n by Proposition 2.12. Thus

$$n - 1 = \dim \mathcal{H}_x \geq \dim \mathcal{B}_x \geq \dim \mathcal{B} + 1 - \dim B \geq n - 1$$

and thus the pure dimension of \mathcal{B}_x is $n - 1$ and \mathcal{B}_x is a union of irreducible components of \mathcal{H}_x . Moreover $\dim B = 2n$. Since each component of H_x is a locus of a component of \mathcal{H}_x , it follows that B_x is also a union of irreducible components of H_x . \square

B comes with the rank 1 distribution $G := (TB)^{\perp_F} \subset TX|_B$ as in (2.6). Since $\text{rk } G = 1$, it must be integrable, so locally there exists a leaf Δ_x for general point $x \in B$, that is $T\Delta_x = G|_{\Delta_x}$. In the course of the proof, we will see that the algebraic closure of each leaf of G is a line. For now, we consider $c \in \mathcal{B}$, a general point, and let $C \subset X$ be the corresponding non-standard line. Note that C is smooth by Theorem 1.2, and that by a dimension count C is not tangent to G (there is at most $2n - 1$ dimensional family of algebraic curves tangent to G , while $\dim \mathcal{B} = 3n - 2$, and $n \geq 2$ by Theorem 2.10). We let $\Gamma_c \subset X$ be the union of leaves through (general) points of C :

$$\Gamma_c = \bigcup_{x \in C} \Delta_x$$

Thus Γ_c contains an open subset, which is an analytic submanifold of X of dimension 2 containing a Zariski-open subset of the non-standard line C . Fix a general point $x \in C$ and consider $\mathcal{B}_x \subset \mathcal{H}_x$. Since \mathcal{B}_x is a union of irreducible components of \mathcal{H}_x by Lemma 5.1, it follows that any line $c' \in \mathcal{H}_x$ in a small neighbourhood of c is in \mathcal{B} . Also c is a smooth point of \mathcal{B} and \mathcal{B}_x .

The tangent space of \mathcal{B}_x at c is a linear space of dimension $n - 1$ contained in $T_c\mathcal{H}_x = H^0(N_{C \subset X} \otimes \mathfrak{m}_x)$, with

$$N_{C \subset X} \otimes \mathfrak{m}_x = (TX|_C/TC) \otimes \mathfrak{m}_x \simeq \mathcal{O}_C(-b-1, (-1)^n, 0^{n-2}, b).$$

by Corollary 4.5. Here $\mathfrak{m}_x \simeq \mathcal{O}_C(-1)$ is the ideal sheaf of the point $x \in C \simeq \mathbb{P}^1$. Under the isomorphism $TX|_C = \mathcal{O}_C(-b, 0^n, 1^{n-2}, 2, b+1)$, the distribution $TB|_C$ is $(TX|_C)^{\geq 0} = \mathcal{O}_C(0^n, 1^{n-2}, 2, b+1)$ by Lemma 4.4 and the distribution $G|_C = (TB|_C)^{\perp_F} = \mathcal{O}_C(b+1)$ (if $b = 1$, then fix the splitting $\mathcal{O}_C(2^2) = \mathcal{O}(2, b+1)$ in such a way that $G|_C$ is the second summand). Thus $\mathcal{O}_C(b+1) \subset N_{C \subset X}$ is the image of $G|_C$ under $TX|_C \rightarrow N_{C \subset X}$. Therefore $\dim T_c\mathcal{H}_x = (n-2) + b+1$ and inside there we have $H^0(G|_C)$ of dimension $b+1$ and $T_c\mathcal{B}_x$ of dimension $n-1$. Thus the two subspaces must intersect (it is possible to argue, that their intersection is of dimension 1, but we can avoid using this). Therefore, G determines a non-trivial distribution \tilde{G} on \mathcal{B}_x . Let $\tilde{\Gamma}_c \subset \mathcal{B}_x$ be an analytically local integral curve of this distribution through c .

We claim that Γ_c and $\bigcup_{c' \in \tilde{\Gamma}_c} C' \subset X$ generically agree: both are 2-dimensional, tangent spaces of both contain G , thus they are covered by leaves of G . They also contain C , which is generically transversal to G , thus both are swept by leaves through points of C . Define $\overline{\Gamma}_c$ to be the Zariski closure of Γ_c . Note that potentially, $\overline{\Gamma}_c$ is of dimension higher than 2, but we will see this is not the case.

Lemma 5.2. *Any two points $x, y \in \overline{\Gamma}_c$ are connected by a contact line from \mathcal{B} contained in $\overline{\Gamma}_c$.*

Proof. We will say that $\text{conn}(Z_1, Z_2)$ holds for subsets $Z_i \subset \overline{\Gamma}_c$, if for all $x \in Z_1$ and $y \in Z_2$, the points x and y are connected by a line from \mathcal{B} contained in $\overline{\Gamma}_c$. First note, that if $\text{conn}(Z_1, Z_2)$ holds, then $\text{conn}(\overline{Z_1}, \overline{Z_2})$ also holds. Indeed, the set of lines contained in $\overline{\Gamma}_c$ is Zariski closed and so the set of $y \in \overline{\Gamma}_c$ such that $\text{conn}(x, y)$ holds for a fixed x is also Zariski closed. Swapping the roles of x and y we get the claim.

Suppose $x \in C$ is a general point and $y \in \Gamma_c$ is in a small neighbourhood of C . Since Γ_c and $\bigcup_{c' \in \tilde{\Gamma}_c} C' \subset X$ agree, hence there is $c' \in \tilde{\Gamma}_c \subset \mathcal{B}_x$, such that $y \in C'$. Thus C' is the required line, and therefore $\text{conn}(C, \overline{\Gamma}_c)$ holds. Now swap the roles of C and C' . Note that $\Gamma_{c'} = \Gamma_c$ (they are both swept out by the same leaves), at least in some neighbourhoods. Thus $\text{conn}(\overline{\Gamma}_c, C')$ holds for any c' near c , so $\text{conn}(\overline{\Gamma}_c, \Gamma_c)$ holds. Taking the Zariski closure, we obtain the desired $\text{conn}(\overline{\Gamma}_c, \overline{\Gamma}_c)$. \square

We conclude from Theorem 1.7 and Lemma 5.2:

Corollary 5.3. *The normalisation of $\overline{\Gamma}_c$ is \mathbb{P}^k .*

□

Let $\mu : \mathbb{P}^k \rightarrow \overline{\Gamma}_c$ be the normalisation map, and consider the distribution $\mu^*G \subset T\mathbb{P}^k$. Consider $\tilde{C} := \mu^{-1}(C)$. Note that $\tilde{C} \simeq \mathbb{P}^1$ is a line in \mathbb{P}^k and it is mapped isomorphically onto C . We have three distributions in $TX|_{\tilde{C}}$: $T\mathbb{P}^k|_{\tilde{C}}$, $G|_{\tilde{C}}$, $T\tilde{C}$. The first one contains the latter two and the latter two are generically transversal to each other. We claim that $G|_{\tilde{C}}$ as a distribution in $T\mathbb{P}^k|_{\tilde{C}}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(1) \subset \mathcal{O}_{\mathbb{P}^1}(1^{k-1}, 2)$. To see that divide out by $T\tilde{C} \simeq \mathcal{O}_{\mathbb{P}^1}(2)$ and recall that the image of $G|_{\tilde{C}}$ in $TX|_{\tilde{C}} = \mathcal{O}_{\tilde{C}}(-b, 0^n, 1^{n-2}, 2, b+1)$ is the $\mathcal{O}_{\mathbb{P}^1}(b+1)$ component. Then, possibly after a change of splitting, the derivative map restricted to \tilde{C} :

$$N_{\tilde{C} \subset \mathbb{P}^k} \simeq \mathcal{O}_{\mathbb{P}^1}(1^{k-2}) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \rightarrow N_{C \subset X} \simeq \mathcal{O}_{\mathbb{P}^1}(-b, 0^n, 1^{n-2}, b+1)$$

is given by embedding $\mathcal{O}_{\mathbb{P}^1}(1^{k-2})$ into $\mathcal{O}_{\mathbb{P}^1}(1^{n-2})$, and the remaining component $\mathcal{O}_{\mathbb{P}^1}(1)$ is mapped non-trivially into $\mathcal{O}_{\mathbb{P}^1}(b+1)$. Therefore $G|_{\tilde{C}} \subset T\mathbb{P}^k|_{\tilde{C}}$ must correspond exactly to this component $\mathcal{O}_{\mathbb{P}^1}(1)$, again possibly after adjusting the splitting $T\mathbb{P}^k|_{\tilde{C}} = N_{\tilde{C} \subset \mathbb{P}^k} \oplus T\tilde{C}$.

We obtained a rank 1 distribution $\mu^*G \subset T\mathbb{P}^k$ such that its restriction to a general line $\mathbb{P}^1 \subset \mathbb{P}^k$ is $\mathcal{O}_{\mathbb{P}^1}(1) \subset T\mathbb{P}^k|_{\mathbb{P}^1}$. Such a distribution can only be obtained as tangent to lines through some fixed distinguished point $y \in \mathbb{P}^k$, see Lemma 2.3. In particular, the leaves of G are algebraic and thus $k = 2$.

It follows that B is dominated by a family of linear subspaces of dimension 2 as defined in §2.1. Let \mathcal{P} be this family and $\mathcal{U}_{\mathcal{P}}$ be the universal bundle:

$$\begin{array}{ccc} & \mathcal{U}_{\mathcal{P}} & \\ \pi_{\mathcal{P}} \nearrow & & \searrow \phi_{\mathcal{P}} \\ \mathcal{P} & \xrightarrow{v} & B. \end{array}$$

The fibres of $\pi_{\mathcal{P}}$ are \mathbb{P}^2 and the image of a general fibre $\pi_{\mathcal{P}}(\mathbb{P}^2) \subset B$ is the surface $\overline{\Gamma}_c$ for some line $c \in \mathcal{B}$. The restriction of $\pi_{\mathcal{P}}$ to $\mathbb{P}^2 \rightarrow \overline{\Gamma}_c$ is the normalisation map. The rational section $v : \mathcal{P} \dashrightarrow \mathcal{U}_{\mathcal{P}}$ is the distinguished point y of general \mathbb{P}^2 constructed above. Let $Y \subset B$ be the closure of the image of $\phi_{\mathcal{P}} \circ v : \mathcal{P} \dashrightarrow \mathcal{U}_{\mathcal{P}} \rightarrow B$, that is the set of all distinguished points.

Lemma 5.4. *The locus Y of distinguished points is not equal to B .*

Proof. Let $y \in B$ be the general point and suppose by contradiction $y \in Y$, that is, there exists \mathbb{P}^2 in \mathcal{P} with the distinguished point mapped to y . Since y is general, G is defined at y . The images of lines in \mathbb{P}^2 through the distinguished point form a one dimensional family of contact lines tangent

to G . In particular, they share the tangent direction at y . This is impossible by Lemma 2.1. \square

For $y \in Y$, let \mathcal{P}^y be the closure of the preimage $(\phi_{\mathcal{P}} \circ v)^{-1}(y)$, that is, essentially, the set of the planes \mathbb{P}^2 with y as the distinguished point.

Lemma 5.5. *For $y \in Y$, let P^y be the locus of \mathcal{P}^y in B , i.e. the union of $\overline{\Gamma}_c$ corresponding to the points in $\mathcal{P}^y \subset \mathcal{P}$. Then $\dim Y = n$ and for a general $y \in Y$, the locus P^y is a component of B_y , whose normalisation is a \mathbb{P}^n .*

Proof. We need to count the dimensions and relative dimensions of the spaces appearing in our construction. Firstly, a general $\overline{\Gamma}_c$ in \mathcal{P} is uniquely determined by a general line $c \in \mathcal{B}$. Thus we have a dominant rational map $\mathcal{B} \dashrightarrow \mathcal{P}$. The fibres are two dimensional: two lines $c, c' \in \mathcal{B}$ determine the same $\overline{\Gamma}_c = \overline{\Gamma}_{c'}$ if and only if the following three conditions are satisfied:

- both C and C' intersect the locus where G is defined;
- both C and C' are transversal to G at their general points;
- $C' \subset \overline{\Gamma}_c$ (assuming the above two conditions, this is equivalent to $C \subset \overline{\Gamma}_{c'}$).

The first two conditions are open in \mathcal{B} . The final one is just a statement that C' is an image of one of the lines in the normalisation $\mathbb{P}^2 \rightarrow \overline{\Gamma}_c$, and there is a two dimensional family of such lines. Thus it follows that:

$$\begin{aligned} \dim \mathcal{P} &= \dim \mathcal{B} - 2 = 3n - 4, \text{ and} \\ \dim \mathcal{P}^y &= 3n - 4 - \dim Y. \end{aligned} \tag{5.6}$$

Apriori, \mathcal{P}^y could be reducible. In such a case Equation (5.6) is about pure dimension: every irreducible component of \mathcal{P}^y has dimension $3n - 4 - \dim Y$.

Let $\mathcal{S} \subset \mathcal{B}$ be the closure of the set of lines tangent to G . Note that $\dim \mathcal{S} = 2n - 1$ since there is a unique line in \mathcal{S} through each general point of B . Also for a general line $c \in \mathcal{S}$ the intersection $C \cap Y$ is non-empty and finite. To prove this claim, the general point $s \in C$ is a general point of B and belongs to a general $\mathbb{P}^2 \in \mathcal{P}$. Thus C is the image of the line in the plane \mathbb{P}^2 , that connects s and the distinguished point of \mathbb{P}^2 . This shows the intersection $C \cap Y$ is non-empty. Also C is not contained in Y , since otherwise $Y = B$ contrary to Lemma 5.4. Thus $C \cap Y$ is finite.

Let $y \in Y$ be a general point and suppose $\mathcal{S}_y \subset \mathcal{S}$ is the set of lines in \mathcal{S} containing y . Then

$$\begin{aligned} \dim \mathcal{S}_y &= \dim \mathcal{S} - \dim Y = 2n - 1 - \dim Y \text{ and} \\ \dim \mathcal{S}_y &\leq \dim \mathcal{B}_y = n - 1, \text{ so that} \\ \dim Y &\geq n. \end{aligned} \tag{5.7}$$

Furthermore, consider the locus $S_y \subset B_y$ of these lines. Note $P^y \subset S_y$, thus:

$$\dim P^y \leq \dim S_y \leq \dim \mathcal{S}_y + 1 = 2n - \dim Y. \quad (5.8)$$

Let $\mathcal{U}_{\mathcal{P}^y}$ be the restriction of $\mathcal{U}_{\mathcal{P}}$ to \mathcal{P}^y , so that the image is $P^y = \phi_{\mathcal{P}}(\mathcal{U}_{\mathcal{P}^y}) \subset B$. We also consider the fibre product

$$\begin{aligned} \mathcal{U}_{\mathcal{P}^y}^2 &:= \mathcal{U}_{\mathcal{P}^y} \times_{\mathcal{P}^y} \mathcal{U}_{\mathcal{P}^y}, \\ \text{so that } \dim \mathcal{U}_{\mathcal{P}^y}^2 &= \dim \mathcal{P}^y + 4 \stackrel{(5.6)}{=} 3n - \dim Y, \end{aligned} \quad (5.9)$$

and its map to $P^y \times P^y$. Less formally, $\mathcal{U}_{\mathcal{P}^y}^2$ is the set of triples $(\mathbb{P}^2, \tilde{u}, \tilde{v})$, with $\tilde{u}, \tilde{v} \in \mathbb{P}^2$ and the triple is mapped to two points u, v in $\overline{\Gamma}_c$ which is the surface normalised by the plane \mathbb{P}^2 . The two points in $\overline{\Gamma}_c$ are the images of \tilde{u} and \tilde{v} under the normalisation map. We claim that the map $\mathcal{U}_{\mathcal{P}^y}^2 \rightarrow P^y \times P^y$ is generically finite onto its image. More precisely, the map is generically finite onto the image of each irreducible component of $\mathcal{U}_{\mathcal{P}^y}^2$.

To prove the claim, suppose that there is a curve $Z \subset \mathcal{U}_{\mathcal{P}^y}^2$ contracted to a single point $(u, v) \in P^y \times P^y$. Suppose moreover, that Z contains a general point z_0 of $\mathcal{U}_{\mathcal{P}^y}^2$ (more precisely, z_0 is a general point of any of the components). More precisely, suppose $(\mathbb{P}_{z_0}^2, \tilde{u}_{z_0}, \tilde{v}_{z_0}) \in Z$ such that:

- If $\tilde{y}_{z_0} \in \mathbb{P}_{z_0}^2$ is the distinguished point, then $\tilde{u}_{z_0}, \tilde{v}_{z_0}, \tilde{y}_{z_0}$ are not on a line in $\mathbb{P}_{z_0}^2$.
- G is defined at u_{z_0} .

Then Z determines a curve in \mathcal{P}^y , such that each \mathbb{P}_z^2 on this curve contains all three of the points y, u, v . In particular, we can take the family of lines connecting u and v on each of the planes \mathbb{P}^2 . By Lemma 2.1, the family of lines must be constant. This is a contradiction, since each plane \mathbb{P}^2 is uniquely determined by the line (because G is defined at u , so in particular, it is defined at a general point of that line).

Thus $\mathcal{U}_{\mathcal{P}^y}^2 \rightarrow P^y \times P^y$ is generically finite onto its image, and

$$\dim \mathcal{U}_{\mathcal{P}^y}^2 \leq 2 \dim P^y. \quad (5.10)$$

We summarise our dimension counts:

$$3n - \dim Y \stackrel{(5.9)}{=} \dim \mathcal{U}_{\mathcal{P}^y}^2 \stackrel{(5.10)}{\leq} 2 \dim P^y \stackrel{(5.8)}{\leq} 2(2n - \dim Y) \quad (5.11)$$

thus $\dim Y \leq n$ and combining with Inequality (5.7):

$$\dim Y = n. \quad (5.12)$$

Thus we obtain the first claim of the lemma. Moreover, rewriting (5.11):

$$2n = \dim \mathcal{U}_{\mathcal{P}^y}^2 \stackrel{(5.10)}{\leq} 2 \dim P^y \stackrel{(5.8)}{\leq} 2n$$

we obtain an equality in (5.8) and in (5.10):

$$\begin{aligned} \dim P^y &= n \\ \dim \mathcal{U}_{\mathcal{P}^y}^2 &= 2 \dim P^y. \end{aligned}$$

Since the map $\mathcal{U}_{\mathcal{P}^y}^2 \rightarrow P^y \times P^y$ is generically finite onto its image, the dimension count proves that the map is dominant. Equivalently, for two general points in P^y , there exists a \mathbb{P}^2 in \mathcal{P}^y , whose image in P^y contains both points. In particular, there exists a line connecting the two points. Thus the normalisation of P^y is \mathbb{P}^n by Theorem 1.7, and the lemma is proved. \square

Now we can conclude:

Proof of Theorem 1.4. With \mathcal{B} and B as above, we have shown in Lemma 5.5, that B is a divisor covered by linear subspaces of dimension n . We claim there is a unique such linear subspace through a general point of B . To see this we will construct a distribution $G' \subset TB$ such that each linear subspace is a leaf of G' .

Consider the lines tangent to G . They form a family of lines of dimension $2n - 1$, which cover the divisor B . This is the same family as was denoted by \mathcal{S} in the proof of Lemma 5.5. Pick a general such line C . As in Section 4, consider the subbundle $(TX|_C)^+ = \bigoplus_{i=1}^n \mathcal{O}(a_i)$, which is the sum of positive line bundle summands, i.e. $a_i > 0$. Then this bundle has a constant rank n . Since there is a unique such line through a general point of B , the subbundles $(TX|_C)^+$ glue together to a rank n distribution G' in $TX|_B$. Each linear space is swept out by the deformations of C with one point fixed. Thus the tangent space to the linear space at its general point is equal to the fibre of G' .

In particular, there is a unique linear space through a general point of B . Consider the closure of the subset of the Chow variety of B consisting of the linear subspaces of dimension n tangent to G' . Let \mathcal{R} be the normalisation of this subset and let $\mathcal{U}_{\mathcal{R}}$ be the normalisation of the pullback of the universal scheme. Then $\pi: \mathcal{U}_{\mathcal{R}} \rightarrow \mathcal{R}$ with the evaluation map $\xi: \mathcal{U}_{\mathcal{R}} \rightarrow B$ form the family of linear subspaces, with a birational evaluation map. Moreover, we claim below the evaluation is finite.

Let $\mathcal{U}_{\mathcal{R}} \xrightarrow{\beta} \mathcal{U}' \xrightarrow{\alpha} B$ be the Stein factorization of ξ with β being a projective morphism with connected fibres, α a finite morphism, and \mathcal{U}' normal. We claim that β is an isomorphism. To see that, suppose ξ contracts an irreducible closed positive dimensional subvariety $Z \subset \mathcal{U}_{\mathcal{R}}$ to a point $x \in X$.

If Z is contained in some fibre \mathbb{P}^n , then $\xi^*L|_Z$ is ample by our assumption on ξ^*L and trivial by our choice of Z , a contradiction. Thus Z maps onto a closed positive dimensional and irreducible $\tilde{Z} \subset \mathcal{R}$. Let $\mathcal{U}_{\tilde{Z}} \subset \mathcal{U}_{\mathcal{R}}$ be the restricted family. The image $\xi(\mathcal{U}_{\tilde{Z}})$ is an irreducible subset of X , whose every point is connected with x by a line. Therefore $\xi(\mathcal{U}_{\tilde{Z}}) \subset H_x$ and by dimension count $\xi(\mathcal{U}_{\tilde{Z}}) = \xi(\mathcal{U}_{\tilde{z}})$, for any $\tilde{z} \in \tilde{Z}$. In particular, \tilde{Z} is contracted under the morphism from \mathcal{R} to the Chow variety. This is impossible, since the morphism is finite (it is a normalisation of a subset of the Chow variety), and $\dim \tilde{Z} > 0$.

Thus $\xi: \mathcal{U}_{\mathcal{R}} \rightarrow B$ is finite and birational, and hence ξ is the normalisation map of B . On the other hand, the morphism $\pi: \mathcal{U}_{\mathcal{R}} \rightarrow \mathcal{R}$ is equidimensional and projective, and the line bundle ξ^*L is ample, in particular, π -ample. Moreover, a general fibre of π is \mathbb{P}^n , and $\xi^*L|_{\mathbb{P}^n} \simeq \mathcal{O}_{\mathbb{P}^n}(1)$. All of these shows we are in the setting of [AD12, Prop. 4.10], and we conclude that $\mathcal{U}_{\mathcal{R}}$ is the projectivisation of the rank $n + 1$ vector bundle $(\pi_*\xi^*L)^*$. In particular, all fibres $\mathbb{P}^n \subset \mathcal{U}_{\mathcal{R}}$ are the normalisations of their images under ξ . □

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